SOLUTION EXERCISE SHEET 22

Exercise 1. The aim of this exercise is to find a vector field on $\mathbb{R}^2\setminus\{0\}$ with vanishing rotational, which is not a scalar potential.

Remark 0.1. We first recall that by results seen in class we know that a continuous vector field f defined on a domain $D \subseteq \mathbb{R}^2$ is a scalar potential if and only if for every closed piecewise C^1 path γ we have that $\int_\gamma f(s) \cdot ds = 0$. Further, f being a scalar potential implies that $\operatorname{rot} f = 0$ and $\operatorname{rot} f = 0$ implies by Green-Riemann that f is a scalar potential if D is simply connected.
We also observe that $\operatorname{rot} f = 0$ is a local property but having a scalar potential on a given set is a global property. This means that if we have a vector field being locally around each point in D a scalar potential then $\operatorname{rot} f = 0$ but this does not necessary mean that f is a scalar potential on the whole set D.

To construct such a vector field, taking in account the above remark, one notice that we have something similar in complex analysis. Indeed, the map $f(z) = \frac{1}{z}$ is holomorphic on the whole set $\mathbb{C}\setminus\{0\}$ but has only a complex primitive if one excludes a half line, i.e. reduces to a simply connected domain in $\mathbb{C} \setminus \{0\}$. This map has a similar behaviour to what we described above. Now, if we consider the set $\mathbb{C}_-$, then the primitive of f is the principal branch of the complex logarithm, which can be written as

$$\operatorname{Ln}(z) = \ln(|z|) + i \operatorname{Arg}(z),$$

where $\operatorname{Arg}(z)$ is the principal argument, which is the unique value $\theta \in (-\pi, \pi)$ such that $z = |z|e^{\theta i}$. The important observation hear is that the discontinuity of Ln on $\mathbb{R}_-$ comes from the principal argument. Thus, the principal argument can be seen as a map from $\mathbb{R}^2 \setminus \mathbb{R}_- \to \mathbb{R}$ with a gradient which can be extended to a C^∞ map on $\mathbb{R}^2 \setminus \{0\}$. Therefore, defining $f = \nabla \operatorname{Arg}$, we see that $\operatorname{rot} f = 0$ on $\mathbb{R}^2 \setminus \{0\}$ as f is locally a scalar potential but f is not a scalar potential on $\mathbb{R}^2 \setminus \{0\}$ as else Arg would have a continuous extension to $\mathbb{R}^2 \setminus \{0\}$.

This can also be shown explicitly by some computations. We have that Arg is given by

$$z \mapsto \begin{cases} \arctan(\frac{y}{x}) \text{ si } x > 0 \text{ et } y \in \mathbb{R} \\ \frac{\pi}{2} \text{ si } x = 0 \text{ et } y > 0 \\ -\frac{\pi}{2} \text{ si } x > 0 \text{ et } y \in \mathbb{R} \\ \pi + \arctan(\frac{y}{x}) \text{ si } x < 0 \text{ et } y > 0 \\ -\pi + \arctan(\frac{y}{x}) \text{ si } x < 0 \text{ et } y < 0 \end{cases}$$

It's gradient on $\mathbb{R}^2 \setminus \mathbb{R}_-$ is therefore given by

$$\nabla \operatorname{Arg}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)^\top ,$$

which is clearly C^∞ on $\mathbb{R}^2 \setminus \{0\}$. We define f as being the above vector field. Then clearly $\operatorname{rot} f = 0$ as

$$\partial_x f_2 = \frac{|(x,y)|^2 - 2x^2}{|(x,y)|^4}$$
$$\partial_y f_1 = -\frac{|(x,y)|^2 - 2y^2}{|(x,y)|^4}.$$

Thus $\operatorname{rot} f = \partial_x f_2 - \partial_y f_1 = 0$. But f is not a scalar potential on $\mathbb{R}^2 \setminus \{0\}$ as

$$\int_{\partial D(0,1)} f(s) \cdot ds = 2\pi \int_0^1 (\sin^2(2\pi t) + \cos^2(2\pi t))dt = 2\pi.$$

Exercise 2. In this exercise we want to compute $\int_\gamma f(s) \cdot ds$ on the ellipse centred in $(2,0)$ and of radii $\sqrt{2}, \sqrt{3}$, where the vector filed f is given by

$$f(x,y) = \left(ye^{xy} + \frac{y^2}{2x} + \frac{y^2}{2}, y\ln(x) + \frac{x^2}{2} + xe^{x,y} \right)^\top .$$

In order to compute this integral we recall that $\int_\gamma g(s) \cdot ds$, if g is a scalar potential on some open set containing γ. Observe that the gradient of $(x,y) \in \mathbb{R}^2 \mapsto e^{xy}$ is given by $(ye^{xy}, xe^{xy})^\top$ and the gradient of $(x,y) \in \mathbb{R}_{>0} \times \mathbb{R} \mapsto \frac{y^2}{2}\ln(x)$ is given by $(\frac{y^2}{2x}, y\ln(x))$. Thus we get that

$$\int_\gamma f(s) \cdot ds = \frac{1}{2} \int_\gamma g(s) \cdot ds,$$

where we have

$$g(x,y) = \left(y^2, x^2\right)^\top .$$

Lastly, by taking the direct parametrization $t \in [0, 2\pi] \mapsto (2 + \sqrt{2}\cos(t), \sqrt{3}\sin(t)$ we get

$$\int_\gamma g(s) \cdot ds = \int_0^{2\pi} (3\sin^2(t), 4 + 4\sqrt{2}\cos(t) + 2\cos^2(t)) * (-\sqrt{2}\sin(t), \sqrt{3}\cos(t))dt$$
$$= 4\sqrt{6} \int_0^{2\pi} \cos^2(t)dt = 4\sqrt{6} \cdot \pi$$

As $\int_0^{2\pi} \sin^n(x)dx = \int_0^{2\pi} \cos^n(x)dx = 0$ if $n \in 2\mathbb{N}+1$. We infer that $\int_\gamma f(s) \cdot ds = 2\sqrt{6} \cdot \pi$.

Exercise 3. *This exercise is very similar to exercise 3 of exercise sheet 9.*
First, we define the map known as the ϵ-*mollifier*.

$$\phi(x) := \begin{cases} \exp\Big(-\dfrac{1}{1-|x|^2}\Big), & |x| \leq 1, \\ 0, & |x| \geq 1. \end{cases}$$

This map is a $C^\infty(\mathbb{R}^n)$ map with support $\operatorname{supp}\phi = \overline{B_1(0)}$. Indeed, the support of the map is trivial and the smoothness can be seen ad it is the composition of $1-|x|^2$ with the map

$$g(x) := \begin{cases} \exp\Big(-\dfrac{1}{x}\Big), & |x > 0, \\ 0, & |x \leq 0, \end{cases}$$

which is a $C^\infty(\mathbb{R})$. The later was proven in analysis 1. Then, we define $C := \int_{\mathbb{R}^n} \phi$. The standard ϵ-*mollifier* for $\epsilon > 0$ are defined as $\eta_\epsilon := \frac{1}{C\epsilon^n}\phi(\frac{x}{\epsilon})$. These maps have support $\operatorname{supp}\eta_\epsilon = \overline{B_\epsilon(0)}$, are non negative, are $C^\infty(\mathbb{R}^n)$ and such that $\int_{\mathbb{R}^n}\eta_\epsilon = 1$.
Next, let $f \in C_0^0(B_1(0))$ and define $(f * \eta_\epsilon) := \int_{\mathbb{R}^n} f(x-y)\eta_\epsilon(y)dy$. This map is clearly well defined as f, η_ϵ have compact support and are continuous. Further, by a translation we see that $(f*\eta_\epsilon) := \int_{\mathbb{R}^n} f(y)\eta_\epsilon(x-y)dy$. Taking $\epsilon > 0$ small enough, for example $\epsilon < \frac{1}{2}$ it is clear that $f * \eta_\epsilon : \mathbb{R}^n \to \mathbb{R}$ has compact support in $B_2(0)$. Finally, it remains to show that $f * \eta_\epsilon$ has regularity $C^\infty(\mathbb{R}^n)$. But this is done exactly as in exercise 3 of exercise sheet 9, therefore we omit to show the details.